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An Optimal Property of Chebyshev Expansions

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INTRODUCTION

Chebyshev polynomials are extremely popular in numerical analysis. One of their virtues is that expansions of functions in series of Chebyshev polynomials are thought to converge more rapidly than expansions in series of other orthogonal polynomials, and some supporting asymptotic evidence for this belief is presented in Lanczos [*2*]. Our purpose here is to demonstrate that for a certain restricted class of functions, the truncated Chebyshev expansion is best in some fairly large class of Jacobi expansions, and, thus, to provide further solid foundation for the Chebyshev faith.

The remainder of the Introduction is devoted to presenting notation and setting the stage. In Section 1, we make precise the sense in which Chebyshev expansions are best, while Section 2 is given over to various counter-examples to the results in Section 1.

Let $P_k^{(\alpha,\beta)}(x)$ be the Jacobi polynomials with $\alpha,\beta > -1$ (that is, the orthogonal polynomials on $I{:}[-1,1]$ with respect to the weight function $w(\alpha,\beta;x) = (1-x)^\alpha(1+x)^\beta$), normalized in the usual fashion (cf. Szegö [*4*], p. 58)). For each (α,β), (γ,δ) we have

$$P_k^{(\alpha,\beta)}(x) = \sum_{j=0}^{k} b_{jk}(\alpha,\beta;\gamma,\delta) P_j^{(\gamma,\delta)}(x), \tag{1}$$

and we adopt the usage that $b_{jk} = 0$ if $j > k$.

We also adopt the convention that

$$P_k^{(\infty,\infty)}(x) = x^k$$

and admit the values $\alpha = \beta = \infty$, with this definition in mind. If (α,β), (γ,δ) are such that

$$b_{jk}(\alpha,\beta;\gamma,\delta) \geqslant 0, \qquad j = 0,\ldots,k;\ k = 0,1,2,\ldots, \tag{2}$$

we say that *Condition P* holds. It is known that if

(i) $\beta = \delta$ and $\alpha > \gamma$,

or

(ii) $\alpha = \beta$, $\gamma = \delta$ and $\alpha > \gamma$,

then Condition P holds. (See, for example, Askey [*1*] and Rainville [*3*]. Askey also gives some other conditions on the indices which imply Condition P.)

To each $f \in C(I)$ we associate its "Fourier" coefficients

$$f_j(\alpha,\beta) = \frac{1}{h_j(\alpha,\beta)} \int_{-1}^{1} f(x) P_j^{(\alpha,\beta)}(x)\, w(\alpha,\beta;x)\, dx, \tag{3}$$

where

$$h_j(\alpha,\beta) = \int_{-1}^{1} [P_j^{(\alpha,\beta)}(x)]^2\, w(\alpha,\beta;x)\, dx$$

and $j = 0,1,2,\ldots$. (Of course, here we assume $\alpha < \infty$.) We put

$$s_k^{(\alpha,\beta)}(x) = \sum_{j=0}^{k} f_j(\alpha,\beta) P_j^{(\alpha,\beta)}(x),$$

$$r_k^{(\alpha,\beta)}(x) = f(x) - s_k^{(\alpha,\beta)}(x)$$

and

$$R_k(\alpha,\beta) = \| r_k^{(\alpha,\beta)} \|,$$

where $\|\cdot\|$ is the uniform norm on I and $k = 0,1,2,\ldots$. We say $f \in U(\alpha,\beta)$ if, and only if,

$$f(x) = \sum_{j=0}^{\infty} f_j(\alpha,\beta) P_j^{(\alpha,\beta)}(x), \tag{3'}$$

uniformly in I, while $f \in A(\alpha,\beta)$ if, and only if, the series is absolutely convergent for each x in I. In case $\alpha = \beta = \infty$, (3′) is assumed to be the Taylor expansion about the origin, and so $f_j(\alpha,\beta)$ are the Taylor coefficients.

1. Main Results

Our results are based on the following simple

LEMMA. *If $f \in U(\alpha,\beta)$ and $\gamma < \infty$ then*

$$f_j(\gamma,\delta) = \sum_{k=j}^{\infty} f_k(\alpha,\beta)\, b_{jk}(\alpha,\beta;\gamma,\delta).$$

Proof

$$\begin{aligned} f_j(\gamma,\delta) &= \frac{1}{h_j(\gamma,\delta)} \int_{-1}^{1} f(x) P_j^{(\gamma,\delta)}(x)\, w(\gamma,\delta;x)\, dx \\ &= \frac{1}{h_j(\gamma,\delta)} \int_{-1}^{1} \left[\sum_{k=0}^{\infty} f_k(\alpha,\beta) P_k^{(\alpha,\beta)}(x)\right] P_j^{(\gamma,\delta)}(x)\, w(\gamma,\delta;x)\, dx \\ &= \sum_{k=0}^{\infty} f_k(\alpha,\beta) \left[\frac{1}{h_j(\gamma,\delta)} \int_{-1}^{1} P_k^{(\alpha,\beta)}(x) P_j^{(\gamma,\delta)}(x)\, w(\gamma,\delta;x)\, dx\right] \\ &= \sum_{k=j}^{\infty} f_k(\alpha,\beta)\, b_{jk}(\alpha,\beta;\gamma,\delta). \end{aligned}$$

(The term-by-term integration is justified by the uniform convergence.)

An immediate consequence of the lemma is

THEOREM 1. *If* $f \in U(\alpha,\beta), f_k(\alpha,\beta) \geqslant 0$, $k = 0,1,2,\ldots$, *and Condition* P *holds, then*

$$f_j(\gamma,\delta) \geqslant 0, \qquad j = 0,1,2,\ldots.$$

The lemma also leads to the following convergence results, which, although not needed in what follows, is stated here for its own interest.

THEOREM 2. *Suppose*

$$\sum_{k=0}^{\infty} |f_k(\alpha,\beta)|\, P_k^{(\alpha,\beta)}(1) < \infty, \tag{4}$$

$\delta \leqslant \gamma$ *with* $\gamma \geqslant -\frac{1}{2}$ *and Condition* P *holds. Then*

$$f \in U(\gamma,\delta) \cap A(\gamma,\delta).$$

Proof. Since $\gamma \geqslant \delta$ and $\gamma \geqslant -\frac{1}{2}$, we know (Szegö [*4*], p. 166) that

$$\max_{-1 \leqslant x \leqslant 1} |P_j^{(\gamma,\delta)}(x)| = P_j^{(\gamma,\delta)}(1), \qquad j = 0,1,2,\ldots. \tag{5}$$

Condition P now implies that

$$\max_{-1 \leqslant x \leqslant 1} |P_k^{(\alpha,\beta)}(x)| = P_k^{(\alpha,\beta)}(1), \qquad k = 0,1,2,\ldots, \tag{6}$$

and hence, in view of (4) and the Weierstrass M-Test $f \in U(\alpha,\beta)$. As a further consequence of (5), the Theorem will be proved if we can show that the sequence

$$F_m = \sum_{j=0}^{m} |f_j(\gamma,\delta)|\, P_j^{(\gamma,\delta)}(1)$$

is bounded.

Since $f \in U(\alpha,\beta)$, the lemma together with Condition P give

$$F_m = \sum_{j=0}^{m} P_j^{(\gamma,\delta)}(1) \left| \sum_{k=j}^{\infty} f_k(\alpha,\beta)\, b_{jk}(\alpha,\beta;\gamma,\delta) \right|$$

$$\leqslant \sum_{j=0}^{m} P_j^{(\gamma,\delta)}(1) \sum_{k=j}^{\infty} |f_k(\alpha,\beta)|\, b_{jk}(\alpha,\beta;\gamma,\delta).$$

Thus

$$F_m \leqslant \sum_{k=0}^{\infty} |f_k(\alpha,\beta)| \sum_{j=0}^{m} b_{jk}(\alpha,\beta;\gamma,\delta) P_j^{(\gamma,\delta)}(1)$$

$$= \sum_{k=0}^{\infty} |f_k(\alpha,\beta)|\, P_k^{(\alpha,\beta)}(1) < \infty,$$

by (4).

We turn now to our main result.

THEOREM 3. *Suppose*

$$f_k(\alpha,\beta) \geqslant 0, \qquad k > n, \tag{7}$$

$\delta \leqslant \gamma$ *with* $\gamma \geqslant -\frac{1}{2}$, *Condition* P *holds and*

$$\sum_{k=0}^{\infty} f_k(\alpha,\beta) P_k^{(\alpha,\beta)}(1) < \infty. \tag{8}$$

Then

$$R_n(\alpha,\beta) \geqslant R_n(\gamma,\delta). \tag{9}$$

Proof. As we saw in the proof of Theorem 2, the hypotheses of our Theorem imply that (6) holds and hence that

$$R_n(\alpha,\beta) = \sum_{k=n+1}^{\infty} f_k(\alpha,\beta) P_k^{(\alpha,\beta)}(1) = \sum_{k=n+1}^{\infty} f_k(\alpha,\beta) \left[\sum_{j=0}^{k} b_{jk}(\alpha,\beta;\gamma,\delta) P_j^{(\gamma,\delta)}(1) \right]$$

$$= \sum_{j=0}^{\infty} P_j^{(\gamma,\delta)}(1) \left[\sum_{k=n+1}^{\infty} f_k(\alpha,\beta) b_{jk}(\alpha,\beta;\gamma,\delta) \right],$$

the exchange of summations being justified since all summands are positive (cf. Titchmarsh [5], Ch. I).

Hence,

$$R_n(\alpha,\beta) = \sum_{j=0}^{n} P_j^{(\gamma,\delta)}(1) \left[\sum_{k=n+1}^{\infty} f_k(\alpha,\beta) b_{jk}(\alpha,\beta;\gamma,\delta) \right]$$

$$+ \sum_{j=n+1}^{\infty} P_j^{(\gamma,\delta)}(1) \left[\sum_{k=j}^{\infty} f_k(\alpha,\beta) b_{jk}(\alpha,\beta;\gamma,\delta) \right]$$

$$= \sum_{j=0}^{n} P_j^{(\gamma,\delta)}(1) \left[\sum_{k=n+1}^{\infty} f_k(\alpha,\beta) b_{jk}(\alpha,\beta;\gamma,\delta) \right] + R_n(\gamma,\delta),$$

in view of Theorem 1 and the Lemma ((8) implies that $f \in U(\alpha,\beta)$). Finally, (7), Condition P and the fact that $P_j^{(\gamma,\delta)}(1) > 0, j = 0, \ldots, n$, conclude the proof.

Remark 1. If in Theorem 3, in place of Condition P we assume that (i) $\beta = \delta$ and $\alpha > \gamma$, or (ii) $\alpha = \beta > \gamma = \delta$, then we can conclude that equality holds in (9) if, and only if, f is a polynomial of degree $\leqslant n$. For, if $f_m(\alpha,\beta) > 0$ for some $m > n$, then either $b_{0m}(\alpha,\beta;\gamma,\delta) > 0$ or $b_{1m}(\alpha,\beta;\gamma,\delta) > 0$ and so

$$\sum_{k=n+1}^{\infty} f_k(\alpha,\beta) b_{jk}(\alpha,\beta;\gamma,\delta) > 0$$

for either $j = 0$ or $j = 1$.

Remark 2. Consider the ultraspherical case ($\alpha = \beta$; $\gamma = \delta$) and assume

$$(-1)^k f_k(\alpha, \alpha) \geqslant 0; \qquad k > n \tag{10}$$

in place of (7). Then since Theorem 3 can now be applied to $f(-x)$, Theorem 3 remains true for $f(x)$.

2. Some Examples

Theorem 3 is, perhaps, most interesting in the ultraspherical case ($\alpha = \beta$; $\gamma = \delta$) since this family of polynomials includes those bearing the names of Legendre and Chebyshev (both kinds), as well as, in our presentation, Taylor. In the ultraspherical case, Theorem 3 is valid if $\alpha > \gamma \geqslant -\frac{1}{2}$. (Recall that $\gamma = -\frac{1}{2}$ corresponds to the Chebyshev expansion.) We shall next present several examples which show that we cannot dispense with requirements (7) or (10), nor demonstrate that the expansion in Chebyshev polynomials produces the smallest error in the somewhat larger class, $\alpha > \gamma > -1$. We suppress the second index in what follows, since we shall deal only with the ultraspherical case.

Examples

1. Take $f(x) = x^3 - \frac{3}{2}x^2 - x$ and suppose $\alpha = \infty$, $n = 0$. Neither (7) nor (10) holds and

$$\min_{-\frac{1}{2} \leqslant \gamma \leqslant \infty} R_0(\gamma) = R_0(\bar{\gamma})$$

where $\bar{\gamma} \sim -.102$.

2. $f(x) = x^2 + bx$, $\alpha = \infty$, $n = 0$. (7) or (10) holds and Theorem 3 is in force. If $0 < |b| < 2$, we have

$$\min_{-1 < \gamma \leqslant \infty} R_0(\gamma) = R_0(\bar{\gamma})$$

where

$$\bar{\gamma} = \frac{b^2 - 4|b|}{4(1 + |b|) - b^2} - \tfrac{1}{2} < -\tfrac{1}{2}.$$

3. $f(x) = x^3 - x^2$, $\alpha = \infty$, $n = 1$. Now (10) holds, yet

$$\min_{-1 < \gamma \leqslant \infty} R_1(\gamma) = R_1(\bar{\gamma})$$

where

$$-1 < \bar{\gamma} < -\tfrac{1}{2}, \qquad (\bar{\gamma} \sim -.543).$$

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